You-tube: Edward Frenkel; Why We Hate Math
Set Theory

A Set is a collection of objects. The objects are called elements of the set.

Particular sets can be defined by (i) a word description or (ii) a list of its elements or (iii) the use of set-builder notation.

(i) The set of counting numbers from 1 to 10

(ii) \{1,2,3,4,5,6,7,8,9,10\} or \{1,2,3,\ldots,10\}

(iii) \{ n \mid n \text{ is a counting number from 1 to 10} \}

For a set to be useful, it must be well defined; that is, given any object, we must be able to tell whether or not it is an element of the given set.
EVERY DECIMAL NUMBER IS A POINT ON THIS LINE
EVERY POINT ON THIS LINE IS A DECIMAL NUMBER
The set $R$ of all decimals is called the Real Numbers and this line is called the Real Number Line.

Important Note: Every rational number (fraction) can be written as a (repeating) decimal and, hence, is a point in $R$. 
An important set is the one with no elements; we call it the **empty** or **null** set and denote it by \{ \} or by the symbol \( \emptyset \).

The empty set can be described by words: “The set of all green elephants in this room,” or by a listing: \{ \}, or by a set builder: \{ n | n is a counting number between 3 and 4 \}.

**SYMBOLS:** Sets are usually designated by capital letters and their elements by small letters. For example,

Let \( A = \{1,2,3\} \) and \( B = \{1,2,3,4,5\} \) and \( C = \{1,2,3,...\} \).

To indicate an element of a set, we use the symbol \( \in \). Thus, we can write \( 2 \in A \) or write \( n \in B \) to indicate that \( n \) is a some number from 1 to 5. Note that the elements of \( C \) go on forever.

To indicate an object is not an element of a set, we use \( \notin \). For example, \( 6 \notin B \) and for any \( x \) whatsoever, \( x \notin \{ \} \).
$A'$ is called the complement of $A$ and $A' = \{x | x \not\in A\}$

We can observe that every element of $B$ is also an element of $A$. In this case, we say $B$ is a subset of $A$ or $B$ is contained in $A$ and write $B \subset A$.

Is $C$ a subset of $A$? Note: For any set $A$, $\emptyset \subset A$ and $A \subset A$
A union $C$ is a new set: $A \cup C = \{x| x \in A \text{ or } x \in C\}$

Note: Union is like addition; and for any set $A$, $A \cup \emptyset = A$

A intersect $C$ is a new set; $A \cap C = \{x| x \in A \text{ and } x \in C\}$

A minus $C$ is a new set; $A - C = \{x| x \in A \text{ and } x \notin C\}$
Definition: The **(Cross) Product** of two sets A and B, is

\[ A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\} \]

Example: If \( A = \{1,2,3\} \) and \( B = \{x,y\} \), then

\[ A \times B = \{(1,x) \ (2,x) \ (3,x) \ (1,y) \ (2,y) \ (3,y)\} \text{ but } \]

\[ B \times A = \{ (x,1) \ (x,2) \ (x,3) \ (y,1) \ (y,2) \ (y,3)\} \]

Example: The plane is simply \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y)\mid x \text{ and } y \text{ are in } \mathbb{R}\} \) and

3-dimentional space is simply \( \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z)\mid x, y \text{ and } z \text{ are in } \mathbb{R}\} \)
Functions

A function is a rule that assigns each element $x$ of a set $X$ to a unique element $y$ in a set $Y$. We write $y=f(x)$ which is read “$y$ equals $f$ of $x$”

EXAMPLE Let $X$ be the set of all students in this room and let $Y$ be the set of all chairs in this room and let $f$ be the rule that assigns each student to the chair he/she is sitting in.
EXAMPLE Let g be the rule that assigns each number \( x \) to the number \( x^2 + 3 \); i.e., \( g(x) = x^2 + 3 \) so

\[
g(1) = 1^2 + 3 = 4 \quad \text{and} \quad g(-2) = (-2)^2 + 3 = 7
\]

NOTE: We often write this function simply as \( y = x^2 + 3 \)

ANOTHER NOTE: The letters we use make no difference whatsoever; the RULE is what counts. Thus,

\[
u = w^2 + 3, \quad z(s) = s^2 + 3 \quad \text{and} \quad P = Q^2 + 3
\]

are all the same function as g above.
Example The speed $S$ of a free-falling object is a function of time $t$. If $t$ is measured in seconds and $S$ is measured in feet per second, then

$$S = 32t$$

Example We want to make a box out of a piece of cardboard measuring $8 \times 10$ inches by cutting out equal squares from the corners and folding up the sides.

The volume $V$ of the resulting box is a function of $x$ and

$$V = (\text{width})(\text{length})(\text{height})$$

$$= (8-2x) (10-2x) (x) = 80x - 36x^2 + 4x^3$$
Algebra with Functions

Given two functions \( f \) and \( g \), we can add, subtract, multiply and divide them.

Example. Let \( f(x) = x^2 + 2 \) and \( g(x) = 3x \). Then

\[
(f + g)(x) = x^2 + 3x + 2
\]
\[
(f - g)(x) = x^2 - 3x + 2
\]
\[
(fg)(x) = 3x^3 + 6x
\]
\[
(f/g)(x) = \frac{x^2 + 2}{3x}
\]

We can even raise them to powers: \( f^2(x) = (x^2 + 2)^2 = x^4 + 4x^2 + 4 \)

and \( f^{1/2}(x) = \sqrt{x^2 + 2} \)
Composition of Functions

An additional operation with functions that is very important is that of composition.

For any two function $f$ and $g$, the composition of $f$ and $g$ is denoted by $f \circ g$ (read $f$ circle $g$) and defined by

$$f \circ g(x) = f(g(x))$$

Again, let $f(x) = x^2 + 2$ and $g(x) = 3x$. Then

$$f \circ g(x) = f(g(x)) = f(3x) = 9x^2 + 2$$
$$g \circ f(x) = g(f(x)) = g(x^2 + 2) = 3x^2 + 6$$

Note that in this example $f$ composed with $g$ is different from $g$ composed with $f$. 
Groups and Fields

Definition: A group \((G, f)\) is a set \(G\) and a function \(f\) on \(G \times G\) to \(G\) satisfying the following conditions (think of \(f\) as “addition” or “multiplication”: in fact, let’s replace \(f\) by “+”)

1) there is a unique element \(0\) in \(G\), called the \textit{identity}, such that \(0 + y = y + 0 = y\) for all \(y\) in \(G\)
2) for each \(x\) in \(G\) there is a unique \(x'\) in \(G\) such that \(x' + x = x + x' = 0\) \((x'\) is called the \textit{inverse} of \(x\))
3) \(x + (y + z) = (x + y) + z\) \((\textit{associative law})\) for all \(x, y\) and \(z\) in \(G\).

Examples: Integers, Rationals, Reals and finite sets of numbers with “clock arithmetic”
Let’s look at “clock arithmetic.”
If it is 9 am now, then in 5 hours it will be 2 pm; that is, in clock arithmetic, \(9 + 5 = 2\). Similarly, \(6 + 7 = 1\) but \(6 + 2 = 8\) as usual. We say that addition is performed \textit{modular 12} (or “mod” 12); that is, we add numbers and if the sum exceeds 11, we subtract off multiples of 12 until a number less than 12 is obtained. For example, 
\[9 + 7 = 4 \pmod{12}\] and 
\[9 + 9 + 9 = 3 \pmod{12}\] and 
\[7 + 5 = 0 \pmod{12}\]

Let \(G_{12} = \{0, 1, 2, \ldots 11\}\) with addition mod 12. Is it a group? What about \(G = \{1, 2, \ldots 11\}\) with multiplication mod 12?

What about \(G_5 = \{0, 1, 2, 3, 4\}\) with the operation of “addition mod 5?” What about \(G = \{1, 2, 3, 4\}\) with the operation “multiplication mod 5?”
Now let $G$ be a set with two operations, say $+$ and $\cdot$, on $G \times G$ to $G$ such that
1) $\{G, +\}$ is a group and 2) $\{G - \{0\}, \cdot\}$ is also a group. Then $G$ is called a field.

Examples: The rational numbers and the real numbers. Also, $G_p = \{0,1,2,\ldots,p - 1\}$ where $p$ is a prime number; these are called finite number fields and will be important in later work. If $p = 5$, then $G_5 = \{0,1,2,3,4\}$ is a finite number field. The addition and multiplication tables for this $G$ are

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Note: $2^2 = 4$ and $3^2 = 4$. What’s going on?

What about $4^2 = 1$?
Let’s “do some algebra” in $G_{12}$
The concept of a group can be extended to sets whose elements are not numbers. For instance, let $X$ be any non-empty set and let

$$G = \{f \mid f \text{ is a function on } X \text{ to the real numbers } \mathbb{R}\}.$$  

Then $(G, +)$ is a group; the identity is the function $f_0$ that maps every element of $G$ to the number 0 and it is easy to see that the conditions of a group are satisfied:

1) For any $f$, $(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x)$
2) For any $f$, $-f$ is the inverse of $f$ and
3) Addition is associative
An extremely important group for us is the Galois Group of symmetries of geometric objects.

Marcus du Sautoy
Symmetry (from Greek συμμετρία symmetria "agreement in dimensions, due proportion, arrangement")[1] in everyday language refers to a sense of harmonious and beautiful proportion and balance.[2] In mathematics, "symmetry" has a more precise definition.

- An object has reflectional symmetry if there is a line which divides it into two pieces which are mirror images of each other.
- An object has rotational symmetry if the object can be rotated about a fixed point without changing the overall shape.
Symmetry in physics has been generalized to mean invariance—that is, lack of change—under any kind of transformation, for example arbitrary coordinate transformations. This concept has become one of the most powerful tools of theoretical physics, as it has become evident that practically all laws of nature originate in symmetries. In fact, this role inspired the Nobel laureate PW Anderson to write in his widely read 1972 article *More is Different* that "it is only slightly overstating the case to say that physics is the study of symmetry." See Noether's theorem (which, in greatly simplified form), states that for every continuous mathematical symmetry, there is a corresponding conserved physical quantity.
Chapter 2 of the book