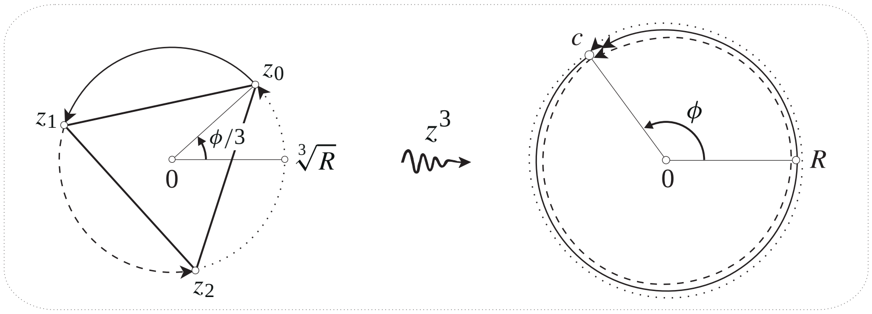

Figure [6]

so the solutions are $z_m = \sqrt[n]{R}\, e^{i(\phi+2m\pi)/n}$. Each time we increase m by 1, z_m is rotated by $(1/n)$ of a revolution (because $z_{m+1} = e^{\frac{2\pi i}{n}} z_m$), producing the vertices of a regular n-gon. Thus the complete set of solutions will be obtained if we let m take any n consecutive values, say $m = 0, 1, 2, \ldots, (n-1)$.

2 Cubics Revisited*

As an instructive application of these ideas, let us reconsider the problem of solving a cubic equation in x. For simplicity, we shall assume in the following that the coefficients of the cubic are all real.

In the previous chapter we saw [Ex. 1] that the general cubic could always be reduced to the form $x^3 = 3px + 2q$. We then found [Ex. 2] that this could be solved using Cardano's formula,

$$x = s + t, \quad \text{where} \quad s^3 = q + \sqrt{q^2 - p^3}, \quad t^3 = q - \sqrt{q^2 - p^3}, \quad \text{and} \quad st = p.$$

Once again, observe that if $q^2 < p^3$ then this formula involves complex numbers.

On the other hand, we also saw [Ex. 3] that the cubic could be solved using Viète's formula:

$$\text{if } q^2 \leq p^3, \text{ then} \quad x = 2\sqrt{p}\cos\left[\tfrac{1}{3}(\phi + 2m\pi)\right],$$

where m is an integer and $\phi = \cos^{-1}(q/p\sqrt{p})$. At the time of its discovery, Viète's "angle trisection" method was a breakthrough, because it solved the cubic (using only real numbers) precisely when Cardano's formula involved "impossible", complex numbers. For a long time thereafter, Viète's method was thought to be entirely different from Cardano's, and it is sometimes presented in this way even today. We shall now take a closer look at these two methods and see that they are really the same.

If $q^2 \leq p^3$, then in Cardano's formula s^3 and t^3 are *complex conjugates*:

$$s^3 = q + i\sqrt{p^3 - q^2} \quad \text{and} \quad t^3 = \overline{s^3} = q - i\sqrt{p^3 - q^2}.$$

These complex numbers are illustrated on the RHS of [7]. By Pythagoras' Theorem,

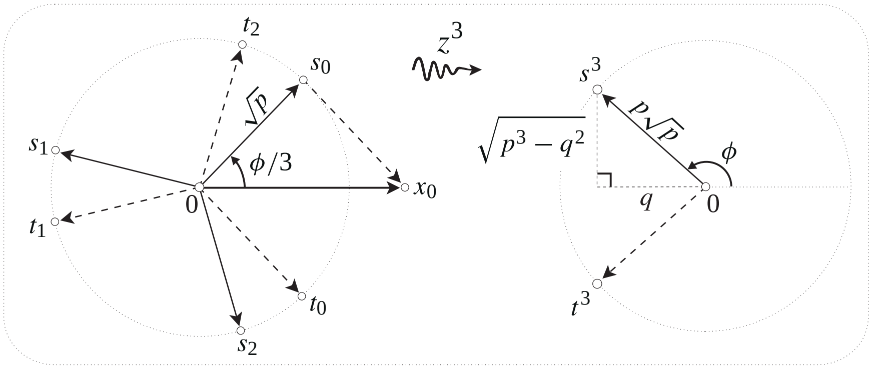

Figure [7]

they both have length $|s^3| = p\sqrt{p}$, and so the angle ϕ occurring in Viète's formula is simply the angle of s^3.

Since s^3 and t^3 lie on the circle of radius $(\sqrt{p})^3$, their preimages under the mapping $z \mapsto z^3$ will lie on the circle of radius $\sqrt{p}$. The LHS of [7] shows these preimages; note that the three values of t are the complex conjugates of the three values of s.

According to the Fundamental Theorem of Algebra, the original cubic should have three solutions. However, by combining each of the three values of s with each of the three values of t, it would seem that Cardano's formula $x = s + t$ yields *nine* solutions.

The resolution lies in the fact that we also require $st = p$. Since p is real, this means s and t must have equal and opposite angles. In the formula $x = s+t$, each of the three values of s must therefore be paired with the conjugate value of t. We can now see how Cardano's formula becomes Viète's formula:

$$x_m = s_m + t_m = s_m + \overline{s_m} = 2\sqrt{p}\,\cos\left[\tfrac{1}{3}(\phi + 2m\pi)\right].$$

In Ex. 4 the reader is invited to consider the case $q^2 > p^3$.

3 Cassinian Curves*

Consider [8a]. The ends of a piece of string of length l are attached to two fixed points a_1 and a_2 in $\mathbb{C}$, and, with its tip at z, a pencil holds the string taut. The figure illustrates the well known fact that if we move the pencil (continuing to keep the string taut) it traces out an *ellipse*, with foci a_1 and a_2. Writing $r_{1,2} = |z - a_{1,2}|$, the equation of the ellipse is thus

$$r_1 + r_2 = l.$$

By choosing different values of l we obtain the illustrated family of confocal ellipses.

In 1687 Newton published his great *Principia*, in which he demonstrated that the planets orbit in such ellipses, with the sun at one of the foci. Seven years earlier,

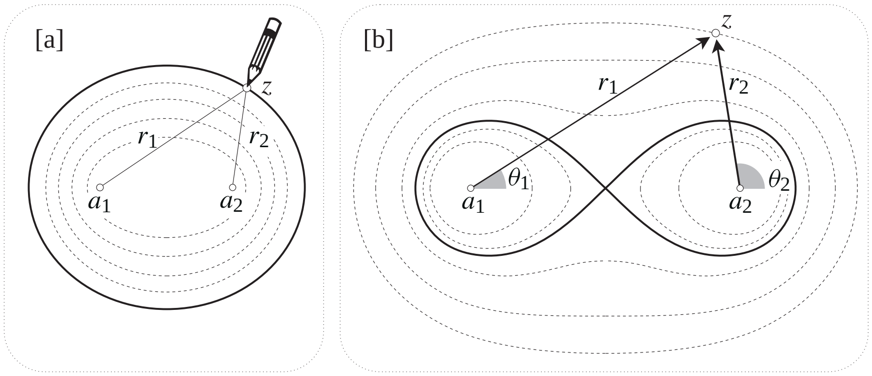

Figure [8]

however, Giovanni Cassini had instead proposed that the orbits were curves for which the *product* of the distances is constant:

$$r_1 \cdot r_2 = const. = k^2. \tag{1}$$

These curves are illustrated in [8b]; they are called *Cassinian curves*, and the points a_1 and a_2 are again called *foci*.

The following facts will become clearer in a moment, but you might like to think about them for yourself. If k is small then the curve consists of two separate pieces, resembling small circles centred at a_1 and a_2. As k increases, these two components of the curve become more egg shaped. When k reaches a value equal to half the distance between the foci then the pointed ends of the egg shapes meet at the midpoint of the foci, producing a figure eight [shown solid]. Increasing the value of k still further, the curve first resembles an hourglass, then an ellipse, and finally a circle.

Although Cassinian curves turned out to be useless as a description of planetary motion, the figure eight curve proved extremely valuable in quite another context. In 1694 it is was rediscovered by James Bernoulli and christened the *lemniscate*—it then became the catalyst in unravelling the behaviour of the so-called *elliptic integrals* and *elliptic functions*. See Stillwell [1989, Chap. 11] and Siegel [1969] for more on this fascinating story.

Cassinian curves arise naturally in the context of complex polynomials. A general quadratic $Q(z) = z^2 + pz + q$ will have two roots (say, a_1 and a_2) and so can be factorized as $Q(z) = (z - a_1)(z - a_2)$. In terms of [8b], this becomes

$$Q(z) = r_1 r_2 \, e^{i(\theta_1 + \theta_2)}.$$

Therefore, by virtue of (1), $z \mapsto w = Q(z)$ will map each curve in [8b] to an origin-centred circle, $|w| = k^2$, and it will map the foci to the origin.

If we follow this transformation by a translation of c, i.e., if we change $z \mapsto Q(z)$ to $z \mapsto Q(z) + c$, then the images will instead be concentric circles centred at $c =$ (image of foci). Conversely, given any quadratic mapping $z \mapsto w = Q(z)$, the preimages of a family of concentric circles in the w-plane centred at c will be the Cassinian curves whose foci are the preimages of c.

In particular, consider the case $c = 1$ and $w = Q(z) = z^2$. The preimages of $w = 1$ are $z = \pm 1$, so these are the foci, and the Cassinian curves are thus centred at the origin. See [9]. Since Q leaves the origin fixed, the lemniscate must be mapped (as illustrated) to the circle of radius 1 passing through the origin. Writing $z = r\, e^{i\theta}$, $w = r^2\, e^{i2\theta}$, and so we see from the figure that the polar equation of the lemniscate is

$$r^2 = 2\cos 2\theta. \tag{2}$$

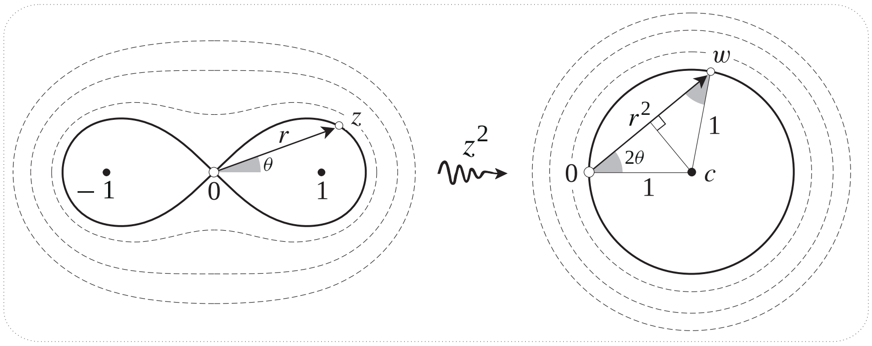

Figure [9]

Returning to [8b], the form of the Cassinian curves may be grasped more intuitively by sketching the modular surface of $Q(z) = (z - a_1)(z - a_2)$. First observe that as z moves further and further away from the origin, $Q(z)$ behaves more and more like z^2. Indeed, since the ratio $[Q(z)/z^2]$ is easily seen [exercise] to tend to unity as $|z|$ tends to infinity, we may say that $Q(z)$ is ultimately equal to z^2 in this limit. Thus, for large values of $|z|$, the modular surface of Q will look like the paraboloid in [3].

Next, consider the behaviour of the surface near a_1. Writing $D = |a_1 - a_2|$ for the distance between the foci, we see [exercise] that $|Q(z)|$ is ultimately equal to Dr_1 as z tends to a_1. Thus the surface meets the plane at a_1 in a cone like that shown in [2]. Of course the same thing happens at a_2.

Combining these facts, we obtain the surface shown in [10]. Since a Cassinian curve satisfies $|Q(z)| = r_1 r_2 = k^2$, it is the intersection of this surface with a plane parallel to $\mathbb{C}$, and at height k^2 above it. As k increases from 0 to a large value, it is now easy to follow the evolution of the curves in [8b] by looking at how this intersection varies as the plane moves upward in [10]. Thus the Cassinian curves may be viewed as a geographical contour map of the modular surface of the quadratic.

Interestingly, Cassinian curves were already known to the ancient Greeks. Around 150 BC, Perseus considered the intersection curves of a *torus* [obtained by rotating a circle C about an exterior line l in its plane] with planes parallel to l. It turns out that if the distance of the plane from l equals the radius of C then the resulting *spiric section of Perseus* is a Cassinian curve. See [11]; in particular,

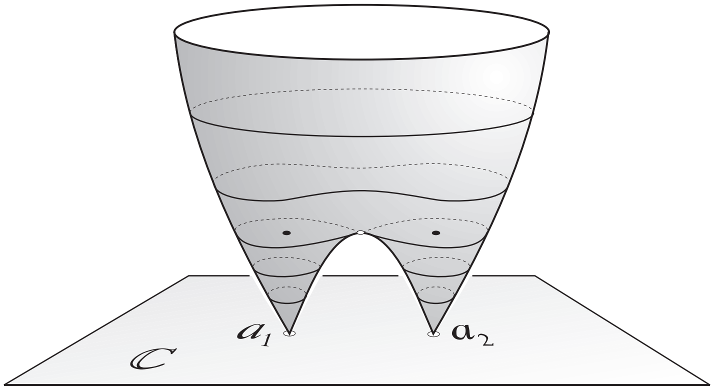

Figure [10]

note how the lemniscate [dashed] makes its surprise appearance when the plane touches the inner rim of the torus. We have adapted this figure from Brieskorn and Knörrer [1986, p. 17], to which the reader is referred for more details.

Returning to the complex plane, there is a natural way to define Cassinian curves with more than two foci: A Cassinian curve with n foci, $a_1, a_2, \ldots, a_n$, is the locus of a point for which the product of the distances to the foci remains constant. A straightforward extension of the above ideas shows that these curves are the preimages of origin-centred circles $|w| = const.$ under the mapping given by the n^{th} degree polynomial whose roots are the foci:

$$z \mapsto w = P_n(z) = (z - a_1)(z - a_2)\cdots(z - a_n).$$

Equivalently, the Cassinian curves are the cross-sections of the modular surface of $P_n(z)$. This surface has n cone-like legs resting on $\mathbb{C}$ at $a_1, a_2, \ldots, a_n$, and for large values of $|z|$ it resembles the axially symmetric modular surface of z^n.

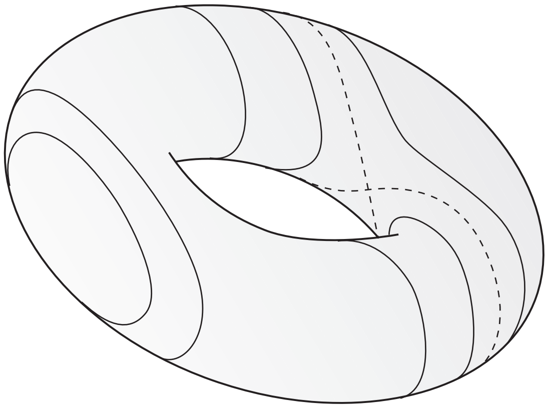

Figure [11]