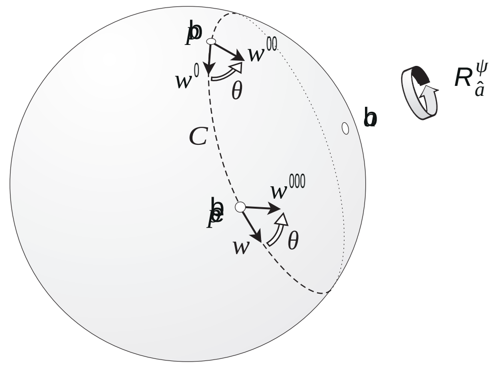

Figure [17]

Deleted Material

III Hyperbolic Geometry

1 The Tractrix and the Pseudosphere

Having studied the intrinsic geometry of surfaces of constant positive Gaussian curvature, we now turn to the intrinsic geometry of surfaces of constant *negative* curvature. Just as there are infinitely many surfaces with $k > 0$, so there are infinitely many with $k < 0$. Beltrami called such surfaces *pseudospherical*. According to the previously stated result of Minding, all pseudospherical surfaces having the same negative value of k possess the same intrinsic geometry. To begin to understand hyperbolic geometry, it is therefore sufficient to examine *any* pseudospherical surface. For our purposes, the simplest one is the pseudosphere, so let us explain how this surface may be constructed.

Try the following experiment. Take a small heavy object, such as a paperweight, and attach a length of string to it. Now place the object on a table and drag it by moving the free end of the string along the edge of the table. You will see that the object moves along a curve like that in [18a], where the Y-axis represents the

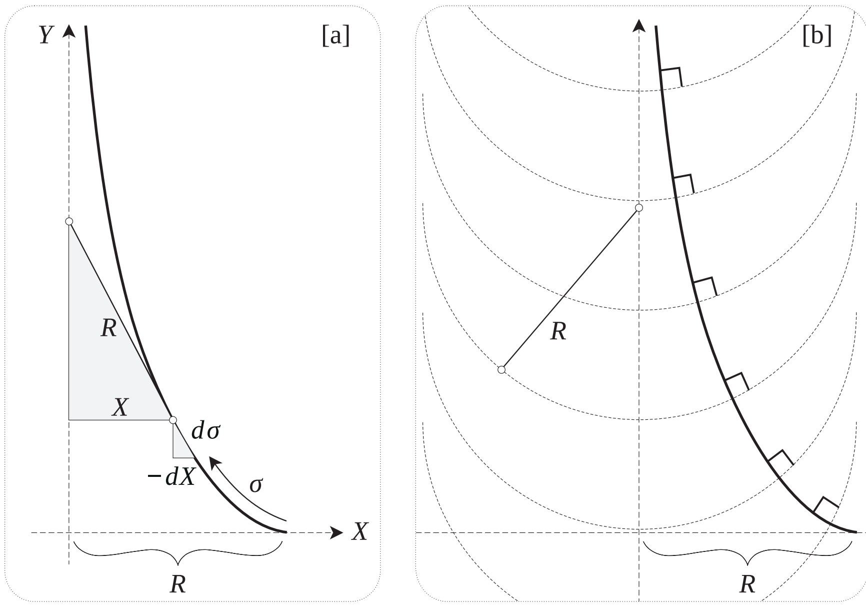

Figure [18]

edge of the table. This curve is called the *tractrix*, and the Y-axis (which the curve approaches asymptotically) is called the *axis*. The tractrix was first investigated by Newton, in 1676.

If the length of the string is R, then it follows that the tractrix has the following geometric property: *the segment of the tangent from the point of contact to the Y-axis has constant length R*. This was Newton's definition of the tractrix. As an interesting aside, it follows [exercise] that the tractrix can be constructed as shown in [18b], namely, as an orthogonal trajectory through the family of circles of radius R centred on the axis. This provides a good method of quickly sketching a fairly accurate tractrix.

Returning to [18a], let σ represent arc length along the tractrix, with $\sigma = 0$ corresponding to the starting position $X = R$ of the object we are dragging. Just as the object is about to pass through (X, Y), let dX denote the infinitesimal change in X that occurs while the object moves a distance $d\sigma$ along the tractrix. From the similarity of the illustrated triangles, we deduce that

$$\frac{dX}{d\sigma} = -\frac{X}{R} \quad \Longrightarrow \quad X = R\, e^{-\sigma/R}. \tag{30}$$

The *pseudosphere of radius R* may now be simultaneously defined and constructed as the surface obtained by rotating the tractrix about its axis. Remarkably, this surface was investigated as early as 1693 (by Christiaan Huygens), two centuries prior to its catalytic role in the acceptance of hyperbolic geometry.

2 The Constant Negative Curvature of the Pseudosphere*

In this optional section we offer a purely geometric proof that the pseudosphere does indeed have constant Gaussian curvature. More precisely, we will use the *extrinsic* definition of κ as the product of the principal curvatures to show that *the pseudosphere of radius R has constant curvature* $\kappa = -(1/R^2)$. Later we will give a purely intrinsic demonstration of this fact, so you won't miss much if you skip the following argument.

Let r and ρ be the two principal radii of curvature of the pseudosphere of radius R. As with any surface of revolution, it follows by symmetry [exercise] that

ρ = *radius of curvature of the generating tractrix;*

r = *the segment of the normal from the surface to the axis;*

as illustrated in [19a]. The problem of determining the Gaussian curvature

$$\kappa = -\frac{1}{r\rho}$$

is thereby reduced to a problem in plane geometry, which is solved in [19b].

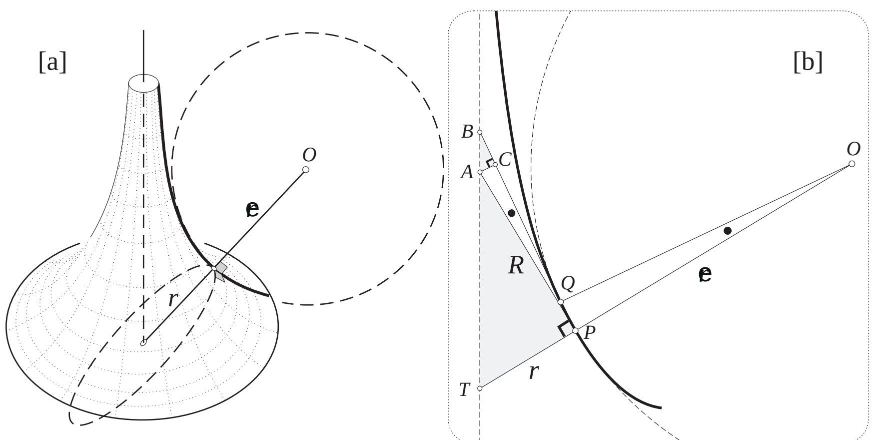

Figure [19]

By definition, the tractrix in this figure has tangents of constant length R. At the neighbouring points P and Q, figure [19b] illustrates two such tangents, PA and QB, containing angle . The corresponding normals PO and QO therefore contain the same angle . Note that AC has been drawn perpendicular to QB.

Now let's watch what happens as Q coalesces with P, which itself remains fixed. In this limit, O is the centre of the circle of curvature, PQ is an arc of this circle, and AC is an arc of a circle of radius R centred at P. Thus,

$$\rho = OP \quad \text{and} \quad \frac{PQ}{OP} = \quad = \frac{AC}{R} \quad \Longrightarrow \quad \frac{AC}{PQ} = \frac{R}{\rho}.$$

Next we appeal to the defining property $PA = R = QB$ of the tractrix to deduce [exercise] that as Q coalesces with P,

$$BC = PQ.$$

Finally, using the fact that as Q coalesces with P the triangle ABC is ultimately similar to the triangle TAP, we deduce that

$$\frac{r}{R} = \frac{AC}{BC} = \frac{AC}{PQ} = \frac{R}{\tilde{\rho}}.$$

Behold!

$$k = -\frac{1}{r\tilde{\rho}} = -\frac{1}{R^2}.$$

3 A Conformal Map of the Pseudosphere

Our next step is to construct a conformal map of the pseudosphere. Recall the benefits of such a map in the case of a sphere: (1) it simultaneously describes all surfaces of curvature $k = +1$; (2) it provides an elegant and practical description of the motions as Möbius transformations. Both of these benefits persist in the present case of negatively curved surfaces; in particular, the (direct) motions of hyperbolic geometry *again* turn out to be Möbius transformations!

For simplicity's sake, *henceforth we shall take the radius of the pseudosphere to be* $R = 1$, so our map will represent pseudospherical surfaces of curvature $k = -1$. As a first step towards a conformal map, [20a] introduces a fairly natural coordinate system (x, σ) on the pseudosphere.

The first coordinate x measures angle around the axis of the pseudosphere, say restricted to $0 \le x < 2\pi$. The second coordinate σ measures arc length along each tractrix generator (as in [18a]). Thus the curves $x = const.$ are the tractrix generators of the pseudosphere [note that these are clearly geodesics], and the curves $\sigma = const.$ are circular cross sections of the pseudosphere [note that these are clearly *not* geodesics]. Since the radius of such a circle is the same thing as the X-coordinate in [18a], it follows from (30) that

> *The radius X of the circle $\sigma = const.$ passing through the point (x, σ) is given by $X = e^{-\sigma}$.*

In our map, let us choose the angle x as our horizontal axis, so that the tractrix generators of the pseudosphere are represented by vertical lines. See [20b]. Thus a point on the pseudosphere with coordinates (x, σ) will be represented in the map by a point with Cartesian coordinates (x, y), which we will soon think of as the complex number $z = x + iy$.

If our map were not required to be special in any way, then we could simply choose $y = y(x, \sigma)$ to be an arbitrary function of x and σ. In stark contrast to

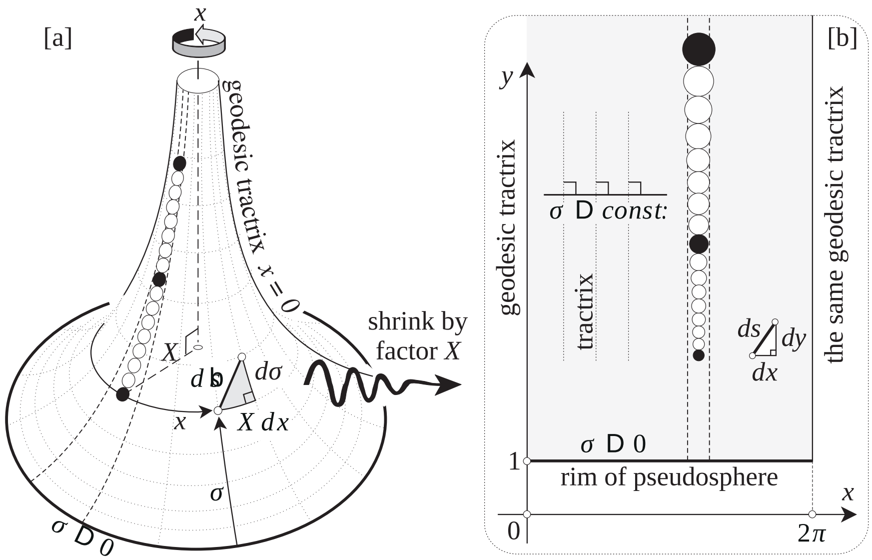

Figure [20]

this, our requirement that the map be *conformal* leaves (virtually) no freedom in the choice of the y-coordinate. Let's try to understand this.

Firstly, the tractrix generators $x = const.$ are orthogonal to the circular cross sections $\sigma = const.$, so the same must be true of their images in our conformal map. Thus the image of $\sigma = const.$ must be represented by a horizontal line $y = const.$, and from this we deduce that $y = y(\sigma)$ must be a function solely of σ.

Secondly, on the pseudosphere consider the arc of the circle $\sigma = const.$ (of radius X) connecting the points (x, σ) and $(x + dx, \sigma)$. By the definition of x, these points subtend angle dx at the centre of the circle, so their separation on the pseudosphere is $X\,dx$, as illustrated. In the map, these two points have the same height and are separated by distance dx. Thus in passing from the pseudosphere to the map, this particular line-segment is shrunk by factor X. [We say "shrunk" because we're dividing by X, but since $X \leq 1$ this is actually an expansion.] However, since the map is conformal, an infinitesimal line-segment emanating from (x, σ) in *any* direction must be multiplied by the *same* factor $(1/X) = e^{\sigma}$. In other words, the metric is

$$d\hat{s} = X\,ds.$$

Thirdly, consider the uppermost black disc on the pseudosphere shown in [20a]. Think of this disc as infinitesimal, say of diameter ϵ. In the map, it will be represented by *another disc*, whose diameter (ϵ/X) may be interpreted more vividly as the angular width of the original disc as seen by an observer on the pseudosphere's axis. Now suppose we repeatedly translate the original disc towards the pseudosphere's rim, moving it a distance ϵ each time. Figure [20a] illustrates the resulting chain of touching, congruent discs. As the disc moves down the pseudosphere, it recedes from the axis, and its angular width as seen from the axis therefore diminishes. Thus the image disc in the map appears to gradually shrink as it moves downward, and the equal distances 8 between the successive black discs certainly do not appear equal in the map.

Having developed a feel for how the map works, let's actually calculate the y-coordinate corresponding to the point (x, σ) on the pseudosphere. From the above observations (or directly from the requirement that the illustrated triangles be similar) we deduce that

$$\frac{dy}{d\sigma} = \frac{1}{X} = e^{\sigma} \quad \Longrightarrow \quad y = e^{\sigma} + const.$$

The standard choice of this constant is 0, so that

$$y = e^{\sigma} = (1/X).$$

Thus the entire pseudosphere is represented in the map by the shaded region lying above the line $y = 1$ (which itself represents the pseudosphere's rim), and the metric associated with the map is

$$d\hat{s} = \frac{ds}{y} = \frac{\sqrt{dx^2 + dy^2}}{y}. \tag{31}$$

For future use, also note that an infinitesimal rectangle in the map with sides dx and dy represents a similar infinitesimal rectangle on the pseudosphere with sides (dx/y) and (dy/y). Thus the apparent area $dx\,dy$ in the map is related to the true area dA on the pseudosphere by

$$dA = \frac{dx\,dy}{y^2}. \tag{32}$$

4 Beltrami's Hyperbolic Plane

In the Introduction we gave the impression that Beltrami had succeeded in interpreting abstract hyperbolic geometry as the intrinsic geometry of the pseudosphere. This is really not possible, and it is *not* what Beltrami claimed.

The abstract hyperbolic geometry discovered by Gauss, Bolyai, and Lobachevsky is understood to take place in a *hyperbolic plane* that is exactly like the Euclidean plane, *except* that lines within it obey the hyperbolic axiom (3):

> *Given a line L and a point p not on L, there are at least two lines through p that do not meet L.*

The constant negative curvature of the pseudosphere ensures that it faithfully represents all consequences of this axiom that deal only with a finite region of the hyperbolic plane. An example of such a consequence is the theorem that the an-